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Particle oscillations in external chaotic fields

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We review here the development of the general formalism for the study of fermion propagation in the presence of stochastic media. This formalism allows the systematic derivation of evolution equations for averaged quantities as survival probabilities and higher order distribution moments. The formalism equally applies to any finite dimensional Schrödinger equation in the presence of a stochastic external field. New integrodifferential equations valid for finite correlated processes are obtained for the first time. For the particular case of exponentially correlated processes a second order ordinary equation is obtained. As a consequence, the Redfield equation valid for Gaussian delta-correlated noise is rederived in a simple way: it has been obtained directly and as the zero-order term of an asymptotic expansion in the inverse of the correlation length. The formalism, together with the quantum correlation theorem is applied to the computation of higher moments and correlation functions. It is shown that equal and unequal time correlators follow similar differential equations.

1. INTRODUCTION

We review here the general formalism for the study of neutrino propagation in the presence of stochastic media [1]. This formalism will allow the systematic derivation of evolution equations for averaged quantities as survival probabilities and higher order moments of them. New integrodifferential equations valid for finite correlation processes are obtained for the first time. For exponentially correlated processes a second order ordinary equation is obtained as a consequence. The Redfield equation valid for Gaussian delta-correlated noise is rederived in a simple way as a particular case: it can be obtained directly or as the zero-order term of an asymptotic expansion in the inverse of the correlation length. In the context of neutrino oscillations, it is shown that the presence of matter noise induces the appearance of a effective complex part in the matter density. Finally the formalism, together with the quantum correlation theorem, will be applied to the problem of the computation of distribution higher moments and non-equal-time correlation functions.

The formalism may be applied to any quantum system governed by similar Schrödinger equations: equations where a stochastic function appears multiplicatively in some part of the Hamiltonian. For simplicity of notation, in this work we will always deal with Hamiltonians which are finite dimensional operators in some Hilbert space, the equations will be obviously of the same type for general systems described by infinite-dimensional Hamiltonians.

2. THE GENERAL CASE

Let us consider a general system whose evolution is described by the following linear random Schrödinger equation:

$$i\partial_t X \quad = \quad \rho(t)L(t)X, \quad X(0) = X_0, \tag{1}$$

where X is a vector (i.e., a multiflavor wave function) or a matrix (i.e., a density matrix) of arbitrary dimension. Equation (1) is defined in an interaction representation, any additive non-stochastic term has been solved for and absorbed in the definitions of $X(t)$ and $L(t)$. $L(t)$ is a general linear operator. It can be an ordinary matrix if X represents a wave function. If X is a density matrix, $L(t)$ is a commutator:

$$L_A(t)X \equiv [A(t), X]. \tag{2}$$

Any other linear operator is admissible: obviously it is always possible to define an enlarged vector space where the action of L is represented by a matrix.

We will assume that $\rho(t)$ in Eq.(1) is a scalar Gaussian process completely determined by its first two moments, which, without loss of generality can be taken as

$$\langle \rho(t) \rangle = 0, \qquad \langle \rho(t)\rho(t') \rangle = f(t,t'). \tag{3}$$

In other terms, $\rho(t)$ is characterized completely by the measure

$$[d\rho] \exp -\frac{1}{2}\int_{-\infty}^{\infty} \rho(t) f^{-1}(t,t')\rho(t')dtdt'. \tag{4}$$

An important particular case is when the process is δ-correlated, the correlation function is then of the form

$$f(t,t') = \Omega^2 \delta\left(\mid t - t' \mid\right). \tag{5}$$

A convenient way of parametrizing a correlation function with a finite correlation length is to use an exponential function:

$$f(t,t') = \Omega^2 \epsilon \exp\left(-\epsilon \mid t - t' \mid\right). \tag{6}$$

In this case the correlation length is defined as $\tau = 1/\epsilon$. The expression (5) is reobtained letting $\tau \to 0$ or $\epsilon \to \infty$ in Eq.(6).

$X(t)$, the solution to Eq.(1), is a stochastic function. The objective of this work is to obtain equations for its ensemble average and higher moments in a systematic way . Let us remark that for us Eq.(1) is purely phenomenological, we assume that it is the result of a more complete microscopic analysis which can account for the randomness of $\rho(t)$ (see for example, Refs.[8,11, 19,20]).

To obtain a differential equation for the average of X we make use of the following well known property: For any Gaussian process $\rho(t)$ characterized by a δ-correlation function (6) and any functional $F[\rho]$ we have the following relation [12]:

$$\langle F[\rho]\rho(t) \rangle = \Omega^2 \left\langle \frac{\delta F[\rho]}{\delta\rho(t)} \right\rangle. \tag{7}$$

For a Gaussian process with an arbitrary correlation function [Eq. (4)] we have instead the general relation:

$$\langle F[\rho]\rho(t) \rangle = \int d\tau \, \langle \rho(t)\rho(\tau) \rangle \left\langle \frac{\delta F[\rho]}{\delta\rho(\tau)} \right\rangle. \tag{8}$$

Let us consider now the evolution operator U for a particular realization of Eq.(1). By definition,

$$X(t) = U(t, t_0)X_0. \tag{9}$$

The operator $U(t, t_0)$ is a functional of ρ, it has the following formal expression in terms of a time ordered exponential:

$$U(t, t_0) = T \exp\left[-i \int_{t0}^{t} \rho(\tau)L(\tau)d\tau\right]. \tag{10}$$

The functional derivative of U with respect to $\rho(t)$ can be computed by direct methods. By differentiating term by term the series expansion for Eq.(10), the result is

$$\frac{\delta U(t, t_0)}{\delta\rho(\tau)} = -iL(\tau)U(\tau, t_0),\ t < \tau < t_0. \tag{11}$$

In order to obtain a differential equation for $\langle X\rangle$ we observe that

$$i\langle\partial_t X\rangle = i\partial_t\langle X\rangle = L(t)\langle\rho(t)X\rangle. \tag{12}$$

Combining together Eqs. (8),(9),(11) we easily obtain the result that the evolution equation for the ensemble average is in the general case an integrodifferential equation given by

$$i\partial_t\langle X(t)\rangle \;=\; -i\int_0^t dt'\ f(t,t')L(t)L(t')\langle X(t')\rangle,\ \langle X(0)\rangle = X_0. \tag{13}$$

This equation, which is exact and of very general validity, is the equation which we were looking for and one of the main results of the present work. Note that previously integrodifferential equations have been obtained, valid for particular cases or in particular limits, using heuristic *ad hoc* arguments (for example, in Ref. [10]). The derivation of Eq.(13) which has been done here is the rigorous justification for such approaches.

In some notable cases Eq.(13) can be reduced to an ordinary differential equation. In the next sections we will see how it can be reduced to, respectively, first and second order equations for the correlation functions given by Eqs. (5) and (6).

3. THE PARTICULAR δ-CORRELATED CASE

For the particular case where the correlation function is of exponential type, a second order ordinary differential equation can be obtained as we will see below. On the other hand, for the simpler δ-correlated case the evolution equation (13) becomes the ordinary differential equation

$$i\partial_t\langle X(t)\rangle \;=\; -i\Omega^2 L^2(t)\langle X(t)\rangle. \tag{14}$$

Taking L as a commutator, this last equation coincides with the Redfield equation derived by Ref. [3]. Note that the effective "Hamiltonian" appearing in the second part of Eq. (14) is not Hermitic anymore (an example of a fluctuation-dissipation effect).

In cases of interest for the neutrino oscillation problem, the original equation for the density matrix is of the slightly simpler form

$$i\partial_t X \;=\; \left[H_0(t) + \rho(t)g(t)H_1, X\right],\ \ X(0) = X_0, \tag{15}$$

where we have written the commutator explicitly. $\rho(t)$ is a stochastic function as before and $g(t)$ an arbitrary scalar function. H_0, H_1 are Hamiltonian matrices, the former contains the average part of $\rho(t)$: $H_0 \equiv H_0'(t) + \rho_0(t) H_1$, with $\rho_0 = \langle \rho \rangle$. The latter is assumed to be time independent.

For the problem described by Eq. (15) the corresponding Redfield equation for the averaged density matrix is of the form

$$\begin{aligned} i\partial_t \langle X \rangle &= [H_0(t), \langle X \rangle] - i\Omega^2 g^2(t) [H_1, [H_1, \langle X \rangle]], && (16) \\ &\equiv H_0^- \langle X \rangle - \langle X \rangle H_0^+ + 2i\Omega^2 g^2(t) H_1 \langle X \rangle H_1, && (17) \end{aligned}$$

where in the last line the following effective Hamiltonians were defined:

$$H_0^\pm = H_0 \pm i g^2(t) H_1^2.$$

The solution of what is called the "coherent" part of Eq. (17) (two first terms of the Hamiltonian) is accomplished by defining the average evolution operator:

$$\langle U^\pm \rangle = T \exp \int d\tau H_0^\pm(\tau), \quad \langle U^- \rangle = \langle U^+ \rangle^\dagger . \tag{18}$$

The coherent part of the density matrix is then

$$\langle X \rangle_{coh} = \langle U^- \rangle X_0 \langle U^+ \rangle^\dagger . \tag{19}$$

Defining a new "coherent" interaction representation by the relations

$$H_L = \langle U^- \rangle^{-1} H_1 \langle U^- \rangle, \quad H_R = \langle U^- \rangle H_1 \langle U^- \rangle^{-1}, \quad \langle X \rangle_I = \langle U^- \rangle \langle X \rangle_{coh} \langle U^- \rangle^{-1}, \tag{20}$$

we arrive at the equation we were looking for, the resolution of the original equation is equivalent to the resolution of the following one:

$$i\partial_t \langle X \rangle_I = 2i\Omega^2 g^2(t) H_L \langle X \rangle_I H_R. \tag{21}$$

There are some important particular cases where Eq. (14) can be solved or considerably simplified by taking into account the algebraic properties of a specific L (in what follows $k(t)$ is always a scalar function).

(a) Let us assume that L is such that $L^2(t) = k(t) L(t)$. This case appears in the computation of the average wave function with matter density noise. Equation (14) reduces to

$$i\partial_t \langle X \rangle = -i\Omega^2 k(t) L(t) \langle X \rangle.$$

The averaged equation is similar to the original one, the nonrandom part of the density is "renormalized" acquiring an imaginary term

$$\rho \to \rho_0 - i\Omega^2 k.$$

This is the density which will appear in the coherent effective Hamiltonians $H_0^\pm$.

(b) The case where $L^2(t) = k(t) I$ with I the identity matrix appears in the computation of the averaged neutrino wave function under noisy magnetic spin-flavor precession. The resulting equation can trivially be integrated (to be compared with the previous case):

$$\langle X(t) \rangle = \exp \left[-\Omega^2 \int_0^t d\tau k(\tau) \right] X_0.$$

The average wave oscillation is damped by a factor equivalent to the one first calculated by Nicolaidis [2]. A similar damping also appears when computing the average density matrix from

Eq. (21). From these differences of behavior with respect to case (a) it is expected that the presence of magnetic field noise can have some influence even if applied far from any resonance region.

(c) The case where $L^4(t) = -k(t)L^2(t)$ appears in the computation of the average density matrix with matter or magnetic noise. We can obtain in this case the "conservation law"

$$\left[1 - \Omega^2 k(t)\right] L^2(t)\partial_t\langle X(t)\rangle = 0.$$

$L^2(t)$ is not invertible because the operator $L(t)$ has a zero eigenvalue. The previous expression has proved to be of practical importance in some concrete numerical applications [13–15].

The presence of zero eigenvalues distinguishes cases (a) and (b) from (c); it can have consequences in the long term behavior of the respective ensemble averages as is shown elsewhere [16].

4. AN ASYMPTOTIC EXPANSION FOR EXPONENTIALLY CORRELATED SYSTEMS

We will see now how Eq. (14) can be obtained as a limiting case when the correlation length tends to zero. For this purpose we use an exponential correlation function as Eq. (6), the integrodifferential evolution equation becomes in this case

$$i\partial_t\langle X(t)\rangle \quad = \quad -i\Omega^2 \,\epsilon \exp(-\epsilon t) \int_0^t dt' \exp(\epsilon\tau) L(t)L(t')\langle X(t')\rangle. \tag{22}$$

Let us compute the asymptotic expansion of the second term of Eq. (22) valid for large ϵ; the following expansion is valid for any function $g(t)$:

$$h(\epsilon) \quad \equiv \quad \epsilon \exp(-\epsilon t) \int_0^t d\tau \;\exp(\epsilon\tau) g(\tau) \sim g(t) - \frac{g'(t)}{\epsilon} + \frac{g''(t)}{\epsilon^2} + \cdots \tag{23}$$

Inserting this expression into Eq. (22), we obtain the following expansion in powers of ϵ:

$$i\partial_t\langle X\rangle \quad = \quad -i\Omega^2 L^2(t)\langle X\rangle + i\frac{\Omega^2}{\epsilon} L(t)\partial_t\left(L(t)\langle X\rangle\right) + o\left(\frac{1}{\epsilon^2}\right). \tag{24}$$

To leading order in $1/\epsilon$, we recover the expression corresponding to the δ-correlated case. At next-to-leading order we get finite-correlation correction terms

$$i\partial_t\langle X\rangle \quad = \quad -i\Omega^2 L^2(t)\langle X\rangle + i\frac{\Omega^2}{\epsilon} L(t)L'(t)\langle X\rangle + i\frac{\Omega^2}{\epsilon} L^2(t)\partial_t\langle X\rangle \tag{25}$$

or, equivalently,

$$\left[1 - \frac{\Omega^2}{\epsilon} L^2(t)\right] \partial_t\langle X\rangle = \left[-\Omega^2 L^2(t) + \frac{\Omega^2}{\epsilon} L(t)L'(t)\right] \langle X\rangle. \tag{26}$$

Finally, We get the following differential equation valid to order $1/\epsilon$, making the assumption that the operator which multiplies the left term is invertible:

$$\partial_t\langle X\rangle \quad = \quad \left(-\Omega^2 L^2(t) + \frac{\Omega^2}{\epsilon} L(t)L'(t) + \frac{\Omega^4}{\epsilon} L^4(t)\right) \langle X\rangle. \tag{27}$$

This equation can be used for finite, but relatively large, correlation lengths. We see that, up to this degree of approximation, not only the ratio level of noise to correlation length (Ω^2/ϵ) is important. We have different regimes according to the value of Ω^2. The first term will be more important for a low noise amplitude ($\Omega^2 << 1$). For strong noise ($\Omega^2 >> 1$) the second term, proportional in this case to L^4, will dominate.

5. AN EXACT DIFFERENTIAL EQUATION FOR EXPONENTIALLY CORRELATED SYSTEMS

In contrast with the approximate approach used in the previous section, we can actually derive a simple, ordinary second order differential equation for the case of exponential correlation. Let us assume that the original equation is of the same decomposable type as the one appearing in Eq. (15) but let us include other cases using the general notation

$$i\partial_t X \quad = \quad \left(L_0(t) + \rho(t) g(t) L_1\right) X, \quad X(0) = X_0, \tag{28}$$

with L_0, L_1 general linear operators as before, the latter time independent. In this case Eq. (13) is of the form

$$i\partial_t \langle X \rangle = L_0(t)\langle X \rangle - i\Omega^2 \epsilon L_1^2 e^{-\epsilon t} g(t) \int_0^t dt' g(t') e^{\epsilon t'} \langle X(t') \rangle. \tag{29}$$

Differentiating the equation once and performing some simple algebra we obtain the following ordinary second order differential equation:

$$\begin{aligned} \left[\partial_t - \lambda(t)\right]\left[i\partial_t - L_0(t)\right]\langle X \rangle &= -i\Omega^2 \epsilon g^2(t) L_1^2 \langle X \rangle, \\ i(\partial_t \langle X \rangle)_0 &= L(0)\langle X \rangle_0, \quad \langle X \rangle_0 = X_0, \end{aligned} \tag{30}$$

with

$$\lambda(t) \equiv -\epsilon + g'(t)/g(t).$$

Let us remark that this equation is exact and probably the most important result of this work from a practical point of view.

6. HIGHER ORDER MOMENTS AND THE QUANTUM REGRESSION THEOREM

Second order distribution moments, expressions of the type $\langle X_i X_j \rangle$, or in general, moments of any order, can also be computed using equations similar to Eq. (13). The straightforward procedure is to define products $X_{ij\cdots k} = X_i X_j \cdots X_k$ and write differential equations for them using the constitutive equations for each of the X_i. The resulting equations are of the same type as Eq. (1). The similarity is obvious when one adopts a tensorial notation and defines products of the form $X^{(n)} = X \otimes \cdots \otimes X$. The task of obtaining evolution equations is especially simple within this notation.

Correlators of quantities at different times also appear in the computation of averages of quantities of physical interest, i.e., expected signal rates. In the most simple case expressions of the type $\langle X_i(t+\tau) X_j(t) \rangle$. We will shortly show that the knowledge of equal-time moments and the application of the quantum correlation theorem is sufficient for the computation of this kind of correlators.

Our objective now is the computation of the correlation function appearing in two steps. First we define the generalized density matrix $X^{(2)}$ as the tensorial product of usual density matrices at two different times and energies:

$$X^{(2)}(E_1, E_2; t_1, t_2) \quad \equiv \quad X(E_1, t_1) \otimes X(E_2, t_2). \tag{31}$$

The average of the element $\langle X^{(2)}_{1111} \rangle = \langle X_{11} X_{11} \rangle$ is evidently the probability correlation function we are looking for.

The differential equation for the equal time function $X^{(2)}(E_1, E_2; t, t)$ is obtained from the individual evolution equations for the matrices $X_{1,2} \equiv X(E_{1,2})$ [indices (1,2) label expressions where E_1, E_2 appear respectively]

$$\partial_t X^{(2)} \quad = \quad H_1 X^{(2)} + X^{(2)} H_2 \equiv \rho L X^{(2)}. \tag{32}$$

Equation (32) is a random linear differential equation, linear in the stochastic variable $\rho(t)$. Applying the formalism developed in the previous section, we can immediately write the equation for the ensemble average $\langle X^{(2)} \rangle$ [Eqs. (13–14)]. Once we know the equal time correlator, we obtain the expression for any other pair (t, t') using the quantum regression theorem [18] which reads as follows. For any vector Markov process Y, if the ensemble average of Y fulfills a Schrödinger-like equation of the type

$$\partial_t \langle Y(t) \rangle = G(t) \langle Y(t) \rangle, \tag{33}$$

with $G(t)$ an arbitrary matrix, then the second order correlations will obey the following equation:

$$\partial_\tau \langle Y_i(t+\tau) Y_l(t) \rangle = \sum_j G_{ij}(\tau) \langle Y_j(t+\tau) Y_l(t) \rangle. \tag{34}$$

Note that the quantum regression theorem is in principle not applicable to systems described by the integral equation Eq. (13). Nevertheless it is applicable to problems where the integral equation can be reduced to an ordinary differential equation such as Eq. (30). Such a second order equation can easily be expressed as a first order one by defining the auxiliary pair process $Y = (X,\ \partial_t X)$.

For the simplest case where ρ is a δ correlated process the equation for non-equal time correlators is explicitly given by

$$i\partial_\tau \langle X^{(2)}(t+\tau, t) \rangle \quad = \quad -i\Omega^2 L^2(\tau) \langle X^{(2)}(t+\tau, t) \rangle. \tag{35}$$

The initial condition, $X^{(2)}(t, t)$, is the equal time correlator previously obtained. For exponentially correlated problems second order equations similar to Eq. (30) can immediately be obtained.

7. CONCLUSIONS AND FINAL REMARKS

In conclusion, in this work we have developed the general formalism for the study of neutrino propagation in stochastic media. This formalism has allowed the systematic derivation of evolution equations for averaged quantities as survival probabilities and higher order moments of them. New integrodifferential equations valid for finite correlation processes have been obtained for the first time. For exponentially correlated processes a second order ordinary equation is obtained as a consequence. The Redfield equation valid for Gaussian δ-correlated noise is rederived in a simple way: it has been obtained directly and as the zero-order term of an asymptotic expansion in the inverse of the correlation length.

The formalism can be generalized in an obvious way to obtain the ensemble average of equations of a slightly more general type than Eq. (1), equations of the form

$$i\partial_t X \quad = \quad \sum_i \rho_i(t) L_i(t) X, \quad X(0) = X_0. \tag{36}$$

These equations appear, for example, in the chaotic neutrino magnetic precession. The result for the case where the ρ_i are δ-correlated in time but mutually uncorrelated is simply

$$i\partial_t \langle X(t) \rangle \quad = \quad -i \sum_i \Omega_i^2 L_i^2(t) \langle X(t) \rangle. \tag{37}$$

The generalization to continuous Schrödinger equations in the presence of random potentials or random external forces is also obvious if we leave aside the mathematical differences coming from the appearance of infinite-dimensional Hilbert spaces. The formalism is of general application to any quantum system governed by similar Schrödinger equations: equations where a stochastic function appears multiplicatively in some term of the Hamiltonian.

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